JOURNAL OF APPROXIMATION THEORY 25, 186–188 (1979)

Note

On Summability and Positive Linear Operators

J. J. SWETITS

Department of Mathematics, Old Dominion University, Norfolk, Virginia 23508

Communicated by Oved Shisha

Received August 15, 1977

Quantitative estimates for approximation by positive linear operators are obtained with the use of a summability method which includes both convergence and almost convergence.

Korovkin's famous theorem [5] regarding convergence of sequences of positive linear operators in the space of continuous functions was put into a quantitative form by Shisha and Mond [8]. In [4] it was shown that Korovkin's results are valid if convergence is replaced by almost convergence, and the modified results were recently put into quantitative form by Mohapatra [7]. It is the purpose of this note to bring some unification through the use of a summability method introduced by H. T. Bell [1].

Let $B = \{A^{(n)}\} = \{(a_{kj}^{(n)})\}$ be a sequence of infinite matrices such that $a_{kj}^{(n)} \geqslant 0$ for $k, j, n = 1, 2, \ldots$. A sequence of real numbers, $\{x_j\}$, is said to be B summable to L if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$$

uniformly in $n = 1, 2, \ldots$.

If, for some matrix A, $A^{(n)} = A$ for $n = 1, 2, \ldots$, then B summability is just matrix symmabitlity by A. If, for $n = 1, 2, \ldots$, $a_{kj}^{(n)} = 1/k$ for $n \leqslant j < k + n$, and $a_{kj}^{(n)}$ is 0 otherwise, then B summability reduces to almost convergence [6]. We also note that the method of order summability of Jurkat and Peyerimhoff [2, 3] is a special case of B summability [1].

Let $\{L_j\}$ be a sequence of positive linear operators from $C[a, b]$ to $C[a, b]$ and let $\{A^{(n)}\} = B$ be a sequence of infinite matrices with non-negative real entries. For $f \in C[a, b]$, $A^{(n)}(f, x)$ denotes the double sequence

$$A_k^{(n)}(f, x) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f(t), x), \qquad k, n = 1, 2, \ldots.$$

0021-9045/79/020186-03$02.00/0

We define $\| A_k(f)\|$ to be

$$\sup_n \sup_{x\in[a,b]} | A_k^{(n)}(f, x)|$$

and we assume that

$$\| A_k(C_0)\| < \infty \tag{1}$$

where $C_0(x) = 1$ for all $x \in [a, b]$. It then follows that, for $f \in C[a, b]$, $\{L_j(f)\}$ is B summable to f, uniformly on $[a, b]$, if and only if

$$\| A_k(f) - f \| \equiv \sup_n \sup_{x\in[a,b]} | A_k^{(n)}(f) - f(x)|$$

tends to 0 as k tends to ∞.

The proofs of the following theorems, which are similar to the proofs of the corresponding results of [7] and [8], are omitted.

THEOREM 1. *Let $\{L_j\}$ be a sequence of positive linear operators from $C[a, b]$ to $C[a, b]$. Let $B = \{A^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Assume* (1) *is satisfied. Then, for $f \in C[a, b]$ and $k = 1, 2,\ldots,$*

$$\|f - A_k(f)\| \leqslant \|f\| \cdot \| A_k(e_0) - 1 \| + w(\mu_k) \| A_k(e_0) + 1 \|$$

where

$$\mu_k{}^2 = \| A_k((t - x)^2)\|,$$

$$\|f\| = \sup_{x\in[a,b]} |f(x)|,$$

and w denotes the modulus of continuity of f.

Let K be the additive Abelian group of real numbers modulo 2π on which the metric d is defined by

$$d(x, y) = \min\{| x - y |, 2\pi - | x - y |\},$$

for $x, y \in K$, $0 \leqslant x, y \leqslant 2\pi$. Let $C(K)$ denote the set of all continuous, real valued functions on K. For $f \in C(K)$, the modulus of continuity, w, is defined by

$$w(f, \delta) = \sup_{\substack{x,y\in K \\ d(x,y)\leqslant\delta}} |f(x) - f(y)|. \tag{2}$$

THEOREM 2. *Let $\{L_j\}$ be a sequence of positive linear operators from $C(K)$ to $C(K)$. Assume* (1) *holds with $[a, b]$ replaced by K, where $B = \{A^{(n)}\}$*

is a sequence of infinite matrices with nonnegative real entries. Then, for $f \in C(K)$ *and* $k = 1, 2, \ldots,$

$$\| A_k(f) - f \| \leqslant \| f \| \cdot \| A_k(e_0) - 1 \| + w(\mu_k) \| A_k(e_0) + 1 \|$$

where w *is defined by* (2),

$$\| f \| = \sup_{x \in K} | f(x) |,$$

and

$$\mu_k^2 = \| A_k(\sin^2((t - x)/2)) \|.$$

We also note that results analogous to Theorems 3 and 4 of [7] can be obtained for B summability.

References

1. H. T. Bell, Order summability and almost convergence, *Proc. Amer. Math. Soc.* **38** (1973), 548–552.
2. W. B. Jurkat and A. Peyerimhoff, Fourier effectiveness and order summability, *J. Approximation Theory* **4** (1971), 231–244.
3. W. B. Jurkat and A. Peyerfimhoff, Inclusion theorems and order summability, *J. Approximation Theory* **4** (1971), 245–262.
4. J. P. King and J. J. Swetits, Positive Linear Operators and Summability, *J. Austral. Math. Soc.* **11** (1970), 281–290.
5. P. P. Korovkin, "Linear Operators and Approximation Theory," Gordon & Breach, New York, 1960.
6. G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta. Math.* **80** (1948), 167–190.
7. R. N. Mohopata, Quantitative results on almost convergence of a sequence of positive linear operators, *J. Approximation Theory* **20** (1977), 239–250.
8. O. Shisha and B. Mond, The degree of convergence of sequences of linear positive operators, *Proc. Nat. Acad. Sci. USA* **60** (1968), 1196–1200.

Printed in Belgium